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form $ab/(2997a) = \times\times\times\times\times\times/999{,}999{,}999$. Then $b \times 333{,}667 = \times\times\times\times\times\times$. Hence $b = 1$, or 2 and $\times\times\times\times\times\times = 333{,}667$ or 667,334. Both suggested possibilities are readily tested and excluded, by virtue of the fact that the last remainder has six digits prior to carrying down the next zero. Thus for $b = 1$, we have a in the form $\times\times$ and $3a \times 2997$ or $8991a$ as the last quantity to subtract. In other words $(10000 - 8991)a$ is the remainder, or $1009a$, where a is a two-digit number. Even 99 is too small for a, so that $b = 1$ leads to a contradiction as before. The solution then is unique.

II. What is a Calculus?

By J. P. Ballantine, University of Michigan.

There is much room for speculation as to the possible ways in which the calculus can be generalized. With a hope of stimulating other answers, rather than any presumption of closing the question, I suggest the following answer.

Consider the functions (1), which we may call the elementary functions of the Newtonian calculus, or briefly the Newtonian elementary functions:

$$0, \quad 1, \quad \frac{x}{1!}, \quad \frac{x^2}{2!}, \quad \frac{x^3}{3!} \cdots. \tag{1}$$

Notice that the derivative of each of the set is its predecessor. In fact, the process of differentiation may be regarded superficially as the process of replacing any of the above functions as they appear in a linear combination by its predecessor.

Thus, if $F(x)$ is given by the equation:

$$F(x) = a + \frac{bx}{1!} + \frac{cx^2}{2!} + \frac{dx^3}{3!} + \cdots, \tag{2}$$

we may infer (under certain conditions of convergence) that the derivative $F'(x)$ is given by the equation

$$F'(x) = b + \frac{cx}{1!} + \frac{dx^2}{2!} + \cdots. \tag{2'}$$

The object of this interpretation of the process of forming a derivative is that it admits readily of a generalization. One can substitute in place of the functions (1) another set of elementary functions, and if as a set they satisfy certain conditions, one can build up a new calculus based on them. The process of differentiation is not altered, but the geometrical interpretation is. Take for instance the set of functions,

$$0, \quad 1, \quad \frac{x}{1}, \quad \frac{x(x-1)}{1\cdot 2}, \quad \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}\ . \tag{1'}$$

The calculus based on these is the well-known calculus of finite differences.

The derivative in this calculus does not mean the slope of the tangent at a point, but the slope of a secant through two points.

There is another way of looking at the derivative, which turns out to be more general. Consider an arbitrary function $F(x)$ developed by the Maclaurin series (2). The constants $a, b, c, \cdots$ are the values of the successive derivatives for $x = 0$. That is to say, b, c, etc., are the values of the first, second, etc., derivatives. With a little imagination, one could call a function itself its own derivative of order zero, from which it would follow that a is the value of the derivative of order zero.

Let us suppose we wished to base a calculus on the set of elementary functions:

$$f_1(x), f_2(x), f_3(x) \cdots.$$

Let us consider a very particular arbitrary function. Let it be particular in the fact that it is expressible as an infinite series, thus:

$$\phi(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + \cdots.$$

Then in our calculus, the values of the successive derivatives of $\phi(x)$ for $x = 0$ are by definition the constants $a_1, a_2, a_3, \cdots$. The values of the derivatives for other values of x may be defined by a horizontal translation of the function $\phi(x)$. In this way the derivatives may be defined as functions.

One of the properties to demand of a set of elementary functions is that they be linearly independent. Otherwise the derivatives are not uniquely defined.

They must also possess some property which will allow one to obtain the values of the constants without recourse to infinite determinants. Briefly, one must be able to eliminate simultaneously all except a finite number of the constants.

This latter property may manifest itself in a most unexpected manner. In the case of the Newtonian elementary functions it appears in the fact that all of the functions except the second vanish for $x = 0$. By use of this and the geometrical relation between the function and the derivative, one is able to obtain the successive constants.

In the case of the elementary functions of the calculus of finite differences, all after the nth vanish for $x = n - 2$, n any integer larger than one.

The foregoing examples were almost too trivial to mention. It remained for Fourier to find the eliminating property for the set of functions (3), and hence build up a calculus based on them.

$$\begin{array}{lllll} \frac{1}{2}, & \cos x, & \cos 2x, & \cos 3x, & \cdots, \\ & \sin x, & \sin 2x, & \sin 3x, & \cdots. \end{array} \tag{3}$$

It is, that the integral of the product of any two different functions of the set between zero and 2π is zero.

The term analytic, as applied to a function, is now seen in a new light. With a slight change of meaning, in the Newtonian calculus an analytic function could be thought of as a function which is expressible in a certain interval by a Taylor development with or without a remainder. Then we may say that a function is analytic in a particular calculus, if it can be handled in that calculus, i.e., if it can be expanded in terms of the elementary functions.

It is a familiar fact that there are functions expressible in a certain interval by a Fourier expansion, which cannot be developed in a Taylor expansion. In general whether a function is analytic or not depends entirely on the calculus under consideration.

We accept as commonplace the fact that the derivative of the nth derivative is a derivative of order $n + 1$, in much the same manner in which students in the grades accept the commutative and other laws of arithmetic. Now that we are dealing with a generalization of the calculus, our logic demands an investigation of the observance of the "laws of calculus." The particular law referred to could well go under the name of the "cumulative law." Any calculus which is obtained from the first generalization mentioned in this paper will obey the cumulative law. The calculus of Fourier does not.

Between the Newtonian and the finite difference calculus on the one hand and the calculus of Fourier on the other we have the calculus of difference quotients.[1] The elementary functions are:

$$1,\quad \frac{(x-x_1)}{1},\quad \frac{(x-x_1)(x-x_2)}{1\cdot 2},\quad \frac{(x-x_1)(x-x_2)(x-x_3)}{1\cdot 2\cdot 3},\quad \cdots. \tag{4}$$

The quantities x_1, x_2, x_3, $\cdots$ are any preassigned constants. The expansion of any analytic function in this calculus will be an infinite series of arbitrary constants multiplied into the successive functions (4). These arbitrary constants are called difference quotients, owing to their law of formation. In reality, however, they are the derivatives of this calculus. They have a geometrical significance, slightly different from that of the Newtonian calculus. In this calculus the cumulative law does not hold.

In summing, I would say that any equation defining the dependence of a function on one or more independent variables, written in a form containing arbitrary constants, may be regarded as a Taylor development of the function in a calculus. The form of the equation will define the calculus. An infinite calculus may be distinguished from a finite calculus by the number of constants. The calculus will consist of the study of the relation between the function and each of the derivatives, and will deal only with functions expressible by the corresponding Taylor development.

REMARKS BY THE EDITOR.

There are of course a number of ways of developing a calculus of linear functional operations, though it is probably unusual to transfer to a new operation the terminology of differentiation. In fact, it is often a matter of special importance to express a new operator in terms of the known operator $D = d/dx$ rather than to displace the latter as the fundamental element of the calculus.

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[1] "Difference Quotients," this MONTHLY, *1919*, 53–59.

Though the idea is an old one, a good deal of its practical development is due to a group of mid-nineteenth century writers.[1] The more recent scientific statement of the theory from the analytic function standpoint is largely the work of Pincherle and Bourlet.[2]

With these last writers a linear operation may be said briefly to consist in a linear transformation on the coefficients in the Taylor development of an analytic function. Thus $D(a_0 + a_1x + a_2x^2 + \cdots) = b_0 + b_1x + \cdots$, where the b's are obtained by the transformation $b_0 = a_1$, $b_1 = 2a_2$, $\cdots$. For any operator A of this kind the linear property $A\{a_1f_1(x) + a_2f_2(x) + \cdots\} = a_1Af_1(x) + a_2Af_2(x) + \cdots$ will be true, always when the number of f's is finite, and in other cases under certain convergence conditions. Mr. Ballantine's first method gives a particular case of this, namely that in which $Af_{i+1}(x) = f_i(x)$ for the chosen fundamental system of functions.

To discuss the second scheme it is necessary to examine the way in which the derivatives of $\varphi(x)$ are defined for a value of x differing from zero. The definition by "parallel translation" implies that there exist relations $f_i(x+y) = \Sigma_j g_{ij}(x)f_j(y)$, and that the jth derivative of $f_i(x)$ is $g_{i,\,j+1}(x)$. It is clear that such relations do not exist for every set of elementary functions. For instance, if these consist of the even powers of x, no such development can be found for $(x + y)^2$, etc. It is a question therefore whether it is strictly correct to speak of the second scheme as more general than the first.

The first and second methods coincide if the elementary functions satisfy the addition equations $f_i(x + y) = \sum_{k=1}^{k=i} f_{i+1-k}(x)f_k(y)$. This is the binomial formula in the case of the Newtonian system, and is also true for the (equally Newtonian) factorial functions (1′).

The operation given by the second process will be one of Pincherle's linear operations only when the "cumulative" law holds. It is doubtful whether anything is gained by extending the name of successive derivatives to functions *not* obtained by the successive application of any one rule.

It is of course possible that an operation defined under the second scheme in the "cumulative" case might have been defined under the first by using a different system of elementary functions. A large class of linear operators A have the following properties: (i) $A(1) = 0$; (ii) the inverse A^{-1} exists.[3] By (i), A^{-1} involves an undetermined constant of "integration" which may be chosen so as to make $A^{-1}(1)$, $A^{-2}(1)$, $\cdots$ vanish when $x = 0$. Selecting arbitrary constants a_i within a certain field of validity, we may then construct a function $\varphi(x) = \sum_{i=1}^{i=\infty} a_iA^{1-i}(1)$ having at the same time the three properties $A^{i-1}\varphi(0) = a_i$, $B_1{}^{i-1}\varphi(0) = a_i$, and $B_2{}^{i-1}\varphi(0) = a_i$, where B_1 and B_2 are Mr. Ballantine's two operators for the system of functions 1, $A^{-1}(1)$, $A^{-2}(1)$, $\cdots$.

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[1] See, among other works, Carmichael, *Treatise on the Calculus of Operations*, London, 1855.

[2] See especially Pincherle e Amaldi, *Le operazione distributive*, Bologna, 1901, both for a thorough discussion of the whole matter, and for numerous references. A simple account of the theory may be found in Chapter VIII of Hadamard, *La série de Taylor et son prolongement analytique*, Scientia Series, Paris, 1901.

[3] It is supposed that A is defined on a power series, as above. $A^n(1)$ means $A^n(1 + 0x + 0x^2 + \cdots)$, and is also a power series.

Perhaps the most interesting questions raised by the discussion are those concerning the relations between the different schemes, and the extent to which these overlap or supplement one another. In particular, are all cumulative operations obtained under the second plan also obtainable under the first?

RECENT PUBLICATIONS.

REVIEWS.

Plane Trigonometry with Practical Applications. By L. E. DICKSON. Chicago, Benj. H. Sanborn and Co., 1922. 4to. 11 + 176 + 35 pages. Price $1.52.

Extracts from the Preface: "This book introduces at an early stage concrete applications of trigonometry to the elementary parts of navigation and surveying, which are the two simplest exact sciences, as well as to the two elementary topics of physics which are known as composition of forces and refraction of light. There is, too, a full explanation of the theory and construction of a Mercator map, a subject of great importance also in geography. Three separate chapters are devoted to these subjects. The necessary terms and ideas are explained at length and illustrated concretely. We thereby obtain an abundance of simple problems whose importance is so convincing that they cannot fail to arouse real interest. Actual experience with classes has firmly convinced the author that these practical applications offer the best means to drive home the principles of trigonometry and to make the subject truly vital. . . .

"The development of the subject is leisurely and the student is given ample time in which to digest each idea. There are given full and lucid explanations of all new terms and ideas. Lack of the precise knowledge of the mathematical meaning of terms is one of the chief sources of difficulty in the study of mathematics. Various terms which should be already familiar to students are re-defined. On the basis of careful readings both of the manuscript and proof sheets by various experienced teachers in high schools and colleges, it is believed that the presentation is throughout both simple and clear.

"The tables are as simple as possible, and accurate for computation to four significant figures, which are ample for all ordinary practical purposes. It is true that some delicate astronomical measurements justify computations with 5, 6, or 7 place tables; but no new theory is involved. The traverse table, which is necessary for navigation and surveying, is really a systematic list of the sides and angles of all right triangles of moderate size. Its additional headings aid in making the present exposition of navigation much simpler than was possible heretofore. The traverse table is extremely useful in all parts of trigonometry and its applications, partly by relieving the monotony of logarithmic computation, but chiefly for the instantaneous checking of computations.

"The chapters on navigation and surveying are each divided into two parts, this making possible either a brief, wholly untechnical, introduction to those applications, or a fuller treatment."

Contents—Chapter I: Trigonometric functions of acute angles, 1–18; Chapter II: Solution of right triangles by means of tables of the natural functions, 19–25; Chapter III: Traverse table; solution of right triangles by inspection; problems on forces and refraction of light, 26–33; Chapter IV: Logarithms, slide rules, 34–47; Chapter V: Solution of right triangles by logarithms, 48–58; Chapter VI: Navigation: dead reckoning. Part I, the sailings (true course assumed). Part II, finding the true course; compass corrections, 59–81; Chapter VII: Land surveying. Part I, balancing a survey, area (true bearings assumed). Part II, surveying instruments; finding true bearings, 82–102; Chapter VIII: Trigonometric functions of any angle, 103–114; Chapter IX: Solution of oblique triangles, 115–132; Chapter X: Relations between functions of several angles, 133–149; Chapter XI: Graphs of the trigonometric functions and their inverses, radians, 150–167; List and index of formulas, 169–170; Index, including index to definitions, 171–172; Answers to certain of the first five exercises of each set, 173–176; Tables, 1–35.

The numerous textbooks that have been prepared by Professor Dickson, to say nothing of his more technical work, have throughout been characterized by a